Not Prepared for submission to JHEP

Notes on Class S theory

Jiashen Chen

USTC

E-mail: ricepud@mail.ustc.edu.cn

Abstract: In this note I will write what I have learned about Class S theory while learning materials on this topic. I mainly learn this topic through the review [1] and I will follow this review. I tried to work the exercises given in [1] out and add those contexts in this note.

Contents

1 6d $\mathcal{N}=(2,0)$ SCFTs

We first introduce the mother theory of SUSY gauge theories, the 6d SCFT.

1.1 Basics of 6d $\mathcal{N}=(2,0)$ SCFT

1.1.1 6d $\mathcal{N}=(2,0)$ Algebra

The maximal dimension for superconformal algebra is 6 and 6d $\mathcal{N}=(2,0)$ theory can be constructed as a superconfomal theory, it is labeled in a simple laced Lie algebra of ADE type, we denote such theory as $\mathcal{X}(\mathfrak{g})$. Some properties of this theory are as follows

- The theory is constructed by a string theory or M theory brane setup, people has no knowledge about its Lagrangian description.
- $\mathcal{X}(\mathfrak{g})$ has vacua whose IR description is an abelian 6d $\mathcal{N}=(2,0)$ theory of self-dual 2 form gauge fields valued in $\mathfrak{h}/W$.
- 5d $\mathcal{N}=2$ with gauge algebra $\mathfrak{g}$ and coupling constant g_{5d} gives an IR description of $\mathcal{X}(\mathfrak{g})$ on a circle of radius g_{5d}^2.
- $\mathcal{X}(\mathfrak{g})$ admits codimension 2 half-BPS defects labeled by nilpotent orbits in $\mathfrak{g}$ and codimension 4 half-BPS defects labeled by representation of $\mathfrak{g}$.
- The vacua of $\mathcal{X}(\mathfrak{g})$ is parametrized by 5 scalar fields, often denote as $\Phi_I, I=6,\cdots,10$

We recall ourselves of superconformal algebras. Generally, bosonic part of superconformal algebras is the conformal algebra $\mathfrak{so}(2,d)$ together with the R-symmetry algebra, and fermionic part consists spinor transform (as spinor reps.) in $\mathfrak{so}(2,d)$, there is a general classification given by W.Nahm at 1978. Thanks to the accidental isomorphisms in $d=3,4,6$, spinor representation in these dimension are fundamental representation in their isomorphic groups[1].

It is known that SCFTs with more than 16 supercharges do not exist for $d\geq 4$ and free for $d=3$, so we only have $\mathcal{N}\leq 2$ allowed in 6d. In 6d, the minimal spinor representation

[1] To be detail, $\mathfrak{so}(2,3)\cong\mathfrak{sp}(4,\mathbb{R}),\mathfrak{so}(2,4)\cong\mathfrak{su}(2,2),\mathfrak{so}(2,6)\cong\mathfrak{so}^*(8)$

of conformal algerba $\mathfrak{so}(2,6)$ is chiral, and superconformal algebra contains such chiral spinor with same chirality, we only have $\mathcal{N}=(1,0)$, $\mathcal{N}=(2,0)$ but no $\mathcal{N}=(1,1)$. The superconformal algebra is denoted as $\mathfrak{osp}(8^*|2\mathcal{N})$, we focus on $\mathfrak{osp}(8^*|4)$ here. This $\mathfrak{osp}(8^*|4)$ consists of conformal algebra of $\mathfrak{so}^*(8)$ (isomorphic to $\mathfrak{so}(2,6)$ as mentioned) and R-symmetry algebra $\mathfrak{usp}(4)\cong\mathfrak{so}(5)$. It is then natural to see supercharge in representation $(\mathbf{8}_s,\mathbf{4})$ of $\mathfrak{so}(6,2)\times\mathfrak{so}(5)_R$, when decomposing $\mathfrak{so}(6,2)\to\mathfrak{so}(5,1)$ (throw away conformal transformations), gives $(\mathbf{4},\mathbf{4})\oplus(\mathbf{4},\mathbf{4})$.

1.1.2 6d $\mathcal{N}=(2,0)$ SCFT

We begin with this result without showing it here :The low energy theory of 6d $\mathcal{N}=(2,0)$ SCFT $\mathcal{X}(\mathfrak{g})$ is roughly speaking a theory of self-dual two-forms gauge fields for a gauge Lie algebra in ADE. The full theory is labelled by an ADE Lie algebra $\mathfrak{g}$, but it should not be regarded as an ordinary non-abelian gauge theory of two-form fields. On a generic point of its tensor branch, its low-energy description consists of abelian tensor multiplets valued in the Cartan subalgebra of $\mathfrak{g}$, modulo the Weyl group.

We first analysis the self-dual form contents, which only exists in even dimension $k=\frac{d}{2}-1$ (where d is the spacetime dimension and k is the self-dual form degree), for $B_{\alpha_1\cdots\alpha_k}$, the self-dualness is defined by $H=dB,\quad H=*H$, explicitly ($s=0$ for Euclidean and $s=1$ for Lorentizian)

$$H_{\alpha_0\alpha_1\cdots\alpha_k}=(k+1)!\partial_{[\alpha_0}B_{\alpha_1\cdots\alpha_k]}=\pm i^{d/2+s}\epsilon_{\alpha_0\cdots\alpha_k\beta_0\cdots\beta_k}\partial^{[\beta_0}B^{\beta_1\cdots\beta_k]} \tag{1.1}$$

of course only when $\frac{d}{2}+s$ is even then it's real. It's Lorentzian for 2d, Euclidean for 4d and Lorentzian for 6d. For 2d, 4d, it's a real scalar field or an Instanton strength. This self-dual form is denoted as $B_{\alpha\beta}$ in 6d $\mathcal{N}=(2,0)$ theory, the supermultiplet (B,λ,Φ) containing it also involves a spinor λ and scalar Φ.

By compactifying this supermultiplet on a circle, it is proved that a 2 form field and a 1 form field is generated. However using the self-dual condition, the 2 form field can be reconstructed from the 1 form field, which shows the 5d content is an ordinary gauge field A_μ giving a 5d $\mathcal{N}=2$ SYM.

1.2 Stringy Origin of 6d $\mathcal{N}=(2,0)$ SCFT

The 6d $\mathcal{N}=(2,0)$ SCFT are 6d $\mathcal{N}=(2,0)$ little string theory's zero string tension limit. The 6d $\mathcal{N}=(2,0)$ Little string theory is also labeled by a simple laced Lie algebra, and it has occurred in many places in string and brane games, such as

- In the weak coupling limit $g_s\to 0$ of N coincident NS5 branes in Type IIA theory, or N coincident M5 branes in M theory. When bulk d.o.f decouples one gets the 6d $\mathcal{N}=(2,0)$ $\mathfrak{su}(N)$ little string theory.

  In this construction of little string theory, 6d $\mathcal{N}=(2,0)$ SCFT $\mathcal{X}(\mathfrak{su}(N))$ arises as the worldvolume theory of N M5 branes in M theory or N NS5 brane in Type IIA theory, supersymmetric vauca are parametrized by the positions of the branes, so generally $(\mathbb{R}^N)^5/S_N$ and d.o.f's mass are proportional to the distance between the branes. We should note this only gives $\mathfrak{a}_N$ and $\mathfrak{d}_N$ (by orbifolding), but not the exceptionals.

- In the weak coupling limit $g_s \to 0$ of Type IIB string theory on $\mathbb{C}^2/\Gamma,\ \Gamma \subset SU(2)$ (whose resolution determines $\mathfrak{g}$)

  In this construction, the Kähler parameter from $\mathbb{P}^1$ volume parametrize the $\mathcal{X}(\mathfrak{g})$ vacua.

1.3 Defects

We can consider introducing more branes in the M5 brane picture

M5	0	1	2	3	4	5	.	.	.	.	.	$\to$ 6d $\mathcal{N} = (2,0)$ theory
M5'	0	1	2	3	.	.	6	7	.	.	.	$\to$ codim 2 defect
M2	0	1	.	.	.	.	.	.	.	.	10	$\to$ codim 4 defect

In this picture, the intersection of M2 and M5 brane is described by a half-BPS codimension 4 operator, and the intersection of M5' and M5 brane is described by a half-BPS codimension 2 operator. These operators are classified by properties of $\mathfrak{g}$: codimension 2 operators are labeled by nilpotent oribits of $\mathfrak{g}$, codimension 4 operators are labeled by representation of $\mathfrak{g}$.

2 Class S theories from 6d

Now we move on by placing the 6d $\mathcal{X}(\mathfrak{g})$ theory on a Riemann-surface C_2 , which gives a 4d $\mathcal{N} = 2$ theory in general. We now explain some aspects of this theory, which is named class S theory.

2.1 Partial Topological Twist and SUSY reduction

We are familiar to topological twists. Generally, it mixes Lorentz symmetry and R symmetry, since the R symmetry current is in the multiplet with stress-energy tensor, the standard way to denote this is by note how the stress-energy tensor changes.

The (partial) topological twist is defined as follows: consider Lorentz rotation from $\mathfrak{so}(1,3)\times \mathfrak{so}(2)$ belongs to seperating $\mathbb{R}^{1,3} \times \mathbb{R}^2$, and for the block diagonal subalgebra $\mathfrak{so}(2)_R \times \mathfrak{so}(3)_R \subset \mathfrak{so}(5)_R$, we define the twist embedded into $\mathfrak{so}(2) \times \mathfrak{so}(2)_R$, where the total symmetry is

$$\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{twist} \times \mathfrak{so}(3)_R \tag{2.1}$$

the stress-energy tensor changes as

$$T^{\mu\nu}_{twist} = T^{\mu\nu} + \frac{1}{4}(\epsilon^{\mu\rho}\partial_\rho J^\nu_{12} + \epsilon^{\nu\rho}\partial_\rho J^\mu_{12}) \tag{2.2}$$

where J_{12} is the generator of $\mathfrak{so}(2)_R$ and $\epsilon^{\mu\nu} = \delta_4^\mu \delta_5^\nu - \delta_5^\mu \delta_4^\nu$ is the antisymmetric tensor on $\mathbb{R}^2$.

How supercharges behave in this twisting process? We recall that $(\mathbf{4}, \mathbf{4})$ is what we have from $(\mathbf{8}_s, \mathbf{4}) \to (\mathbf{4}, \mathbf{4})$. For each 6d Weyl spinor $\mathbf{4}$ of $\mathfrak{so}(1,5)$, decomposes into a pair of 4d Weyl spinors $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ under $\mathfrak{so}(1,3)$ and these spinors have opposite charges $\pm\frac{1}{2}$ under $\mathfrak{so}(2)$. Also, each $\mathbf{4}$ of $\mathfrak{so}(5)_R$ decomposes into two $\mathbf{2}$ of $\mathfrak{so}(3)_R$ with charges $\pm\frac{1}{2}$ in $\mathfrak{so}(2)_R$. By twisting defined above, the charges added in $\mathfrak{so}(2)$ and $\mathfrak{so}(2)_R$.

$$\begin{aligned}
(\mathbf{4}, \mathbf{4}) &= \left((\mathbf{2}, \mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}\right) \otimes \left(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}\right) \\
&= (\mathbf{2}, \mathbf{1}; \mathbf{2})_{\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{2}, \mathbf{1}; \mathbf{2})_{\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2})_{-\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2})_{-\frac{1}{2},-\frac{1}{2}} \\
&= (\mathbf{2}, \mathbf{1}; \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{1}; \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2})_{-1} \\
&:= Q_z^{\alpha A} \oplus Q^{\alpha A} \oplus \bar{Q}^{\dot{\alpha} A} \oplus \bar{Q}_{\bar{z}}^{\dot{\alpha} A} \quad \alpha, \dot{\alpha}, A = 1, 2
\end{aligned} \tag{2.3}$$

The middle two are scalars, which we want in topological twist. This enables us for general $\mathbb{R}^{1,3} \times C$, we preserve $\mathfrak{so}(1,3)$ Poincare symmetry, $\mathfrak{so}(3)_R = \mathfrak{su}(2)_R$ R symmetry and two supercharges $Q^{\alpha A}$ and $\bar{Q}^{\dot{\alpha} A}$. Which is the 4d $\mathcal{N} = 2$ algebra.

2.2 Coulomb Branch

The Coulomb branch of a 4d $\mathcal{N} = 2$ theory is described by giving a VEV to Coulomb branch operators, which is defined as

$$\bar{Q}^{\dot{\alpha} A} \phi = 0 \tag{2.4}$$

the Coulomb branch $\mathcal{B}$ only depends on the complex structure of C. Generally, vacuum moduli space is a mixed branch $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$, where $\mathcal{C}_\alpha$ is a special Kähler manifold parametrized by Coulomb branch operators, and $\mathcal{H}_\alpha$ is a hyper-Kähler manifold parametrized by Higgs branch operators. As we know in the Seiberg-Witten theory, Coulomb branch encodes the IR behavior of the theory and generally the effective description is a theory with gauge group $U(1)^{\dim \mathcal{B}}$.

2.2.1 Coulomb branch and Coulomb branch operators

What are the Coulomb branch operators now? Recall that the vacua of $\mathcal{X}(\mathfrak{g})$ is parametrized by VEV of scalar fields and the low energy theory has B and λ added, under the $(\mathfrak{so}(1,3) \times \mathfrak{so}(3)_R) \times \mathfrak{so}(2) \times \mathfrak{so}(2)_R$ symmetry algebra, these fields transform as (recall, these subscripts are $\mathfrak{so}(2)$ charges)

$$
\begin{aligned}
&B \in (\mathbf{3},\mathbf{1},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{0,0} \oplus (\mathbf{2},\mathbf{2},\mathbf{1})_{\pm1,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0} \\
&\lambda \in (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\pm\frac{1}{2}} \\
&\Phi_8,\Phi_9,\Phi_{10} \in (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0} \\
&\Phi_z := \Phi_6 + i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,1} \\
&\Phi_{\bar{z}} := \Phi_6 - i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,-1}
\end{aligned}
\tag{2.5}
$$

We can deduce

$$\bar{Q}^{\dot{\alpha}A}\Phi_z = 0 \tag{2.6}$$

since $\bar{Q}^{\dot{\alpha}A} \in (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\frac{1}{2}}$. We have to explain the reason here: When $\bar{Q}^{\dot{\alpha}A}$ has its action on Φ_z, we expect to get a fermion. However, the R charge is $\frac{3}{2} = \frac{1}{2} + 1$ here, but the fermion expects a $\pm\frac{1}{2}$ R charge. This leads to the vanish. More precisely, after the twist there is no fermionic field with the Lorentz and R-symmetry quantum numbers required to appear on the right-hand side of $\bar{Q}^{\dot{\alpha}A}\Phi_z$. Therefore Φ_z is $\bar{Q}$-closed.

The gauge invariant operators vanished by $\bar{Q}^{\dot{\alpha}A}$ are polynomials, which is denoted by $P_k(\Phi_z)$ with degree d_k, $k = 1, \cdots, \mathrm{rank}\mathfrak{g}$ generally, which are Casimirs of the corresponding $\mathfrak{g}$. For $\mathfrak{su}(N)$, this polynomial is $\mathrm{Tr}(\Phi_z^j)$, $j = 2, \cdots, N$ and for $\mathfrak{so}(2N)$, is $\mathrm{Tr}(\Phi_z^j)$, $j = 2, \cdots, 2N-2$[2]. To get a coordinate free object on Riemann surface C, often the operator is translated to holomorphic j-differential $P_j(\Phi_z)dz^j$. This can be covered in to characteristic polynomials, for $\mathfrak{su}(N)$

$$\det(X - \Phi_z dz) = X^N - \sum_{j=2}^{N} \mathcal{O}_j X^{N-j} \tag{2.7}$$

We have $\mathcal{O}_2 = \mathrm{Tr}\left(\frac{\Phi_z^2}{2}\right) dz^2$, $\mathcal{O}_3 = \mathrm{Tr}\left(\frac{\Phi_z^3}{3}\right) dz^3, \cdots$.

VEV of these operators, $\phi_j := \mathcal{O}_j$, gives the coordinate of Coulomb branch. Of course these ϕ_j does not depends on $\mathbb{R}^{1,3}$ coordinates, but how they depend on C coordinates? The anwser is holomorphically:

$$\partial_{\bar{z}}\phi_j = 0 \tag{2.8}$$

besides the holomorphic polynomial understanding, we can physically understand this: since $\{\bar{Q}^{\dot{\alpha}A}, \bar{Q}_{\bar{z}}^{\dot{\beta}B}\} \sim \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\partial_{\bar{z}}$, we have

$$\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\partial_{\bar{z}}\phi_j \sim \left\langle \bar{Q}^{\dot{\alpha}A}(\bar{Q}_{\bar{z}}^{\dot{\beta}B}\mathcal{O}_j)\right\rangle + \left\langle \bar{Q}_{\bar{z}}^{\dot{\beta}B}(\bar{Q}^{\dot{\alpha}A}\mathcal{O}_j)\right\rangle = 0 \tag{2.9}$$

the first term vanishes because the twisted compactification preserves the supercharge $\bar{Q}$ and second by definition of Coulomb branch operators.

[2] j is the Casimir degree for corresponding groups. For E_6, it's $2, 5, 6, 8, 9, 12$. For E_7, it's $2, 6, 8, 10, 12, 14, 18$. For E_8, it's $2, 8, 12, 14, 18, 20, 24, 30$.

Therefore, if C is has no punctures, the Coulomb branch can be written as

$$\mathcal{B} = \bigoplus_{k=1}^{r} H^0(C, K^{\otimes d_k}) \tag{2.10}$$

where K is the canonical bundle, $H^0(C, K^{\otimes d_k})$ denotes the holomorphic differential space with degree d_k and r is the rank of Lie algebra. For $\mathfrak{su}(N)$, it's

$$\mathcal{B} = \bigoplus_{j=2}^{N} H^0(C, K^{\otimes j}) \tag{2.11}$$

and every point on Coulomb branch is given by $(\phi_2, \cdots, \phi_N)$. When C has a puncture, stuff are complicated, we need to further more discuss flavor symmetries and other parameters, which we won't discuss about here.

2.2.2 Hitchin system

We introduce Hitchin field $\varphi = \varphi_z dz$ and formally 4d Coulomb branch is parametrized by $\mathrm{Tr}(\varphi^j)$, this means $\phi_j \sim \langle \mathrm{Tr}\Phi_z^j \rangle \sim \mathrm{Tr}(\varphi_z^j)$. However, when we compactify the theory to 3d, the 3d Coulomb branch is given by the solution of Hitchin system $\mathcal{M}$[2]

$$\begin{aligned} &F + [\varphi, \bar{\varphi}] = 0 \\ &\bar{D}\varphi = 0 \quad D\bar{\varphi} = 0 \end{aligned} \tag{2.12}$$

modding out gauge transformations. The is equivalent to the flatness condition of $A + \varphi_z dz + \bar{\varphi}_{\bar{z}} d\bar{z}$ together with a gauge fixing condition. In this perspective, the 4d Coulomb branch is a base of Hitchin fibration

$$\mathcal{M} \to \mathcal{B} \tag{2.13}$$

which maps $\varphi \to \{P_k(\varphi)\}$.

2.3 Seiberg-Witten curve

Seiberg-Witten curve repackages the data from Coulomb branch into a geometric object. This curve is defined on the canonical line bundle of C, T^*C, whose fiber at a point in C consists of one forms at that point, locally the coordinates (z, x) admits a one form xdz. For $\Sigma \subset T^*C$, the Seiberg-Witten curve is generally defined as

$$\langle \det(x - \Phi_z) \rangle = \det(x - \varphi_z) = x^N - \sum_{j=2}^{N} u_j(z) x^{N-j} = 0 \tag{2.14}$$

where $\phi_j = \langle \mathcal{O}_j \rangle = u_j(z) dz^j$, while the Seiberg-Witten form is $\lambda = xdz$. Note that in this definition, every term transforms as a holomorphic N form, so it's well defined under transformations. Also, locally this is generally a N times equation, which gives a N sheet covering of C.

We can see in a M theory perspective of the Seiberg-Witten curve. Since $\mathcal{X}(\mathfrak{su}(N))$ is the worldvolume theory of M5 branes, the full geometric set up of $\mathcal{X}(\mathfrak{su}(N))$ is to consider M theory on $\mathbb{R}^{1,3} \times T^*C \times \mathbb{R}^3$ and to place M5 brane along the zero section of T^*C ($C \hookrightarrow T^*C$ that maps $z \in C$ to $(z, x = 0)$) times $\mathbb{R}^{1,3}$. Moving to the Coulomb branch, the M5 brane shift to different places $x = x_i(z)$, these branes are indistinguishable so they generically reconnect into an N-sheeted ramified cover $\Sigma \subset T^*C$ of C. The UV curve C defines the theory itself while IR curve/Seiberg-Witten curve Σ characterizes the given Coulomb branch vacua.

We can also consider a 2d surface $D \subset T^*C$ whose boundary lies in SW curve $\partial D \subset \Sigma$, and $D \times \mathbb{R}$ where $\mathbb{R}$ is the time direction. This defines a M2 brane ending on a M5 brane, on the 4d picture, this is a particle sitting still as time passes, its mass is the area of D

$$m = \int_D |dzdx| \geq \left| \int_D dzdx \right| = \left| \int_D d(xdz) \right| = \left| \int_{\partial D} \lambda \right| \tag{2.15}$$

where SW curve gives the BPS bound. This is denoted as $T(\mathfrak{g}, C, D)$.

Let us summarize the basic dictionary of class S theories. Starting from a 6d $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$ and compactifying it on a punctured Riemann surface C, we obtain a 4d $\mathcal{N} = 2$ theory

$$\mathcal{T}_{\mathfrak{g}}[C]. \tag{2.16}$$

The Lie algebra $\mathfrak{g}$ determines the type of the Hitchin system and the possible gauge groups appearing in weakly coupled frames. The Riemann surface C is called the UV curve. Its complex structure moduli are identified with exactly marginal gauge couplings in four dimensions, while different degeneration limits of C correspond to different weak-coupling duality frames.

Punctures on C specify boundary conditions for the Hitchin field and give rise to flavor symmetries in the four-dimensional theory. The mass parameters are encoded in the leading singular parts of the Hitchin field near the punctures. The Coulomb branch is described by the base of the Hitchin fibration, or equivalently by the meromorphic differentials

$$\phi_k(z) = \mathrm{Tr}\, \Phi_z^k, \qquad k \in \{d_i\}, \tag{2.17}$$

where d_i are the degrees of the Casimir invariants of $\mathfrak{g}$. Finally, a pants decomposition of C gives a Lagrangian description whenever such a description exists: three-punctured spheres give matter sectors, while tubes connecting them correspond to gauging diagonal flavor symmetries by 4d $\mathcal{N} = 2$ vector multiplets.

Class S data	4d $\mathcal{N} = 2$ meaning
$\mathfrak{g}$	ADE type and Hitchin system
C	UV curve
complex structure of C	exactly marginal couplings
punctures	flavor symmetries and masses
$\phi_k = \mathrm{Tr}\, \Phi_z^k$	Coulomb branch parameters
pants decomposition	duality frame
three-punctured sphere	matter sector
tube	gauging a diagonal flavor symmetry

(2.18)

2.4 Tubes and tinkertoys

What have we known about the class S theory yet? We only know about how it behaves on IR at some part of the vacua configuration (Coulomb branch). But what the general picture is this 4d theory? To let the description reproduces the description that SW curve does on Coulomb branch, we can move forward to see how the general picture looks like.

2.4.1 Gluing

Recall that the partial topological twist makes the Class S theory only depends on the complex structure of C but not the metric, thus, topologically. We can do topological surgeries to C.

Assuming that $\bar{C}$ (the close covering of C) has punctures $p_1, p_2 \in \bar{C}$, locally, punctured disks are topologically equivalent to semi-infinite cylinders. In terms of complex coordinate z, w on these punctured disks (where p_1 at $w = 0$, p_2 at $z = 0$), we can glue these cylinders by introducing a parameter q

$$zw = q \tag{2.19}$$

since the cylinder are parametrized by $w = e^{s_1 + i\theta_1},\ z = e^{s_2 + i\theta_2}$, then

$$s_1 + s_2 = \log|q| \tag{2.20}$$

indicating $|q|$ controls the length of the tube, and $\arg q$ controls the twist. Since $|q| \to 0$ gives the length goes to $-\infty$, the aspect ratio (length over radius) is $-\frac{\log|q|}{2\pi}$.

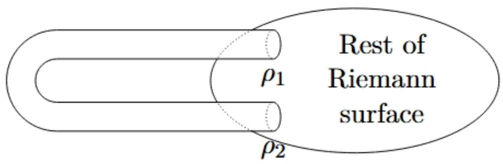

Figure 1.

We can first go closer the four-punctured sphere, consider the cross-ratio generally. If we have the several points $\{w_i\}$ on $\mathbb{P}^1$, we can always use the mapping

$$z(w) = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \tag{2.21}$$

to map three of them to $0, 1, \infty$ respectively. Other coordinates are defined also up to this mapping. We take the four puncture sphere we shall meet at the next section, which the four punctures are $\{0, q, 1, \infty\}$ (explained below), if the original 4 punctures are exchanged, there are six possibilities of how q behaves

$$q,\ 1-q,\ \frac{1}{q},\ \frac{1}{1-q},\ \frac{q}{q-1},\ \frac{q-1}{q} \tag{2.22}$$

it is easy to see that $1-q$ and $\frac{1}{q}$ generates others, which is the T and S duality. We can also consider how gluing two pants by a tube results, for $P_x\{0,1,\infty\}$ and $P_y\{0,1,\infty\}$, we use the identification $y = qx$ (at $1 < |x| < \frac{1}{|q|}$), as the $zw = q$ glued $z = 0$ and $w = 0$ together, this suppose to glue $x \to \infty$ and $y \to 0$ together, resulting the global coordinate $\{0, 1, q, \infty\}$, which is the puncture coordinates of four-punctured sphere. More generally this can be promoted to n-punctured sphere by introducing gluing parameters $q_1, \cdots, q_{n-3}$. The puncture coordinates will be

$$0, q_1 \cdots q_{n-3}, \cdots, q_{n-3}, 1, \infty \tag{2.23}$$

What does this picture do to physics theories? We denote along the tube, the coordinate is x^4 and around the tube is x^5. Locally the tube behaves as $I_{x^4} \times S^1_{x^5}$, by the reduction of $\mathcal{X}(\mathfrak{g})$ on S^1, 5d $\mathcal{N}=2$ theory is obtained, and the coupling is $g^2_{5d} \sim L_5$. Thus the theory on the cylinder is a 5d $\mathcal{N}=2$ theory on a interval I with length

$$L_4 \sim -\frac{\log|q|}{2\pi} \times 2\pi L_5 = -\log|q| L_5 \tag{2.24}$$

we have

$$\frac{1}{g^2_{4d}} = \frac{L_4}{g^2_{5d}} = -\log|q| \tag{2.25}$$

which means the long tube limit equals to the weak coupling limit at 4d. What about the phase of q? When the theory reduce from $\mathcal{X}(\mathfrak{g})$ to 5d $\mathcal{N}=2$, the KK momentum around x^5 corresponds to the 5d instanton charge under the instanton current $J^{inst}_\mu = \epsilon_{\mu\nu\rho\sigma\tau}\mathrm{Tr}(F^{\nu\rho}F^{\sigma\tau})$, and the translation operator P_5 correspond to J^{inst}_4 integrating over $x^0, \cdots, x^3$. When we twist θ during the gluing

$$\theta = \mathrm{Im}\log q \tag{2.26}$$

this correspond a shift on x^5, which is a term

$$\theta \mathrm{Tr}(F \wedge F) \tag{2.27}$$

on 4d. To sum up, q defines the coupling of 4d[3]

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad q \sim e^{2\pi i \tau} \tag{2.28}$$

[3]We should note that $q \sim e^{2\pi i\tau}$ is only valid at the weak coupling limit

2.4.2 Pants decomposition

Any punctured Riemann surface C with genus g and puncture n (except $(0,0),(0,1),(0,2),(1,0)$) can be decomposed into three punctured spheres (pants). For each pants decomposition of C, there is a corresponding cusp in the moduli space $\mathcal{M}_{g,n}$, at this cusp, C is described by three-punctured spheres joined by infinitely thin tubes. Each such tube yields an infinitely weakly coupled vector multiplet in the 4d theory.

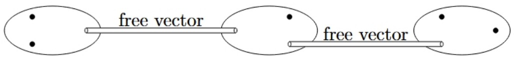

Figure 2.

Recall that punctures are 4d matters, suppose they enjoy a gauged flavor symmetry G by couples to the vector multiplet. When $g \to 0$, the flavor symmetry is restored

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}(F^2) + |D\phi|^2 \xrightarrow{g\to 0} \mathrm{Tr}(d\tilde{A})^2 + |d\phi|^2 \quad \tilde{A} := g^{-1}A \tag{2.29}$$

this means in this cusp, we can expect 6d fields localized on each pair of pants to decouple from each other since the vector multiplet joining them become free.

We arrived as the follow picture, the building block of $T(\mathfrak{g}, C, D)$ are class S theories called tinkertoys associated to three-punctured spheres. These tinkertoys have flavor symmetries associated to each puncture. For each tube, consider the flavour symmetry groups F_1 and F_2 associated to the two punctures that it connnects and gauge a suitable diagonal subgroup $F \subset F_1 \times F_2$ using a 4d $\mathcal{N} = 2$ vector multiplet. This yields $T(\mathfrak{g}, C, D)$, which is formally written as

$$T(\mathfrak{g}, C, D) = \left(\prod_{pants} T(\mathfrak{g}, sphere - 3pt)\right) / \left(\prod_{tubes} gauge\ group\right) \tag{2.30}$$

3 Lagrangians for class S theories

3.1 Trifundamental tinkertoys

The result of this subsection is that there is only one tinkertoys for $\mathfrak{g} = \mathfrak{su}(2)$ with three tame punctures, named T_2, which can be regarded as 4 free hypermultiplets. As we know in the last section, this is one of the main building block of $\mathfrak{su}(2)$ class S theory. Although there is no first principles to derive this, this consist with many checks so people believe this is true.

3.1.1 Tame punctures

We recall Hitchin fields $\varphi(z)dz$ and high order gauge invariant differentials $\mathrm{Tr}(\varphi^{d_k}) \sim dz^{d_k}$. Denoting $\phi_{d_k} \sim dw^{d_k}$ on the tube, mapping the tube to a punctured disk using $z = e^w$

makes

$$\phi_{d_k} \sim \frac{dz^{d_k}}{z^{d_k}} \tag{3.1}$$

this means that a puncture comes from cutting a tube results a d_k pole of d_k order differentials. This motivates the definition of tame punctures : Hitchin field $\varphi(z)dz$ has first order pole on these punctures.

3.1.2 $\mathfrak{su}(2)$ case

For $\mathfrak{su}(2)$, there is a single Casimir $\phi_2 = \frac{1}{2}\mathrm{Tr}(\varphi^2)dz^2$, the Hitchin field is

$$\varphi(z) \sim \left(\frac{\mathrm{diag}(m, -m)}{z - z_i} + O(1)\right) dz \Longrightarrow \phi_2(z) = \left(\frac{m^2}{(z - z_i)^2} + O\left(\frac{1}{z - z_i}\right)\right) dz^2 \tag{3.2}$$

here m is the mass parameter[4]. One thing we need to note is the zero mass limit does not vanish the pole. This is because φ is up to conjugation, where we are free to

$$\begin{pmatrix} m & 0 \\ & -m \end{pmatrix} \sim \begin{pmatrix} m & 1 \\ & -m \end{pmatrix} \tag{3.3}$$

since both of these are nilpotent when $m \to 0$, these means there's only a order 1 pole for ϕ_2 when it's massless. The massless puncture carries a $SU(2)$ flavor symmetry and turning on a constant scalar $\phi_{background} = m$ in a background vector multiplet coupled to that symmetry changes the puncture to the massive one.

The first clue we have for T_2 is it has 4 free hypermultiplets. We have at least $SU(2)^3$ flavor symmetry here for three punctures. For 4 free hypermultiplets, there are $USp(8)$ symmetry (by the commuting part of $O(8)$ and $SU(2)_R$), and $SU(2)^3 = SU(2) \times Spin(4) \subset USp(8)$.

Another way to express these 4 free hypermultiplets is a "trifundamental half-hypermultiplet". For each hypermultiplet are pair of $\mathcal{N} = 1$ chiral multiplet $(q, \tilde{q})$, and in $SU(2)$ fundamental representation. 4 of them can be denote collectively as $q_{a,i,u}$, $a, i, u = 1, 2$, in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ representation. Since there's a pesudo real condition

$$\tilde{q}^{aiu} = \epsilon^{ab}\epsilon^{ij}\epsilon^{uv}q_{bjv} \tag{3.4}$$

this is a half-hypermultiplet in the trifundamental representation. When the background vector multiplet scalars are turned on for $SU(2)^3$, the 8 multiplets have masses choices $\pm m_1 \pm m_2 \pm m_3$, in particular one of the 4 hypermultiplets become massless when $m_2 = \pm m_1 \pm m_3$.

3.1.3 Seiberg-Witten curve of T_2

Another consistency check of the theory is by Seiberg-Witten curve of T_2, we denote by m_1, m_2, m_3 the mass parameters of punctures at $0, 1, \infty$, $\phi_2(z)$ is expected to have second

[4]This is because the Seiberg-Witten curve now is $x^2 = \frac{m^2}{(z-z_i)^2}$, the period of SW differential gives $\pm m$, which is the BPS bound

order poles. The result is that ϕ_2 is unique, for $\phi_2 = u_2(z)dz^2$, the general form respect to holomorphic is

$$u_2(z) = \frac{az^2 + bz + c}{z^2(z-1)^2} \tag{3.5}$$

by setting the poles condition

$$\begin{cases} z \to 0 & u_2(z) \sim \frac{m_1^2}{z^2} \\ z \to 1 & u_2(z) \sim \frac{m_2^2}{(z-1)^2} \\ z \to \infty & u_2(z) \sim \frac{m_3^2}{z^2} \end{cases} \tag{3.6}$$

this gives

$$u_2(z) = -\frac{m_1^2}{z^2(z-1)} + \frac{m_2^2}{z(z-1)^2} + \frac{m_3^2}{z(z-1)} \tag{3.7}$$

as the unique form. As the T_2 only has 4 free hypermultiplet, there is no Coulomb branch for this theory. We can study the SW curve $\Sigma : x^2 = u_2(z)$ to know its properties at IR, for Σ is a ramified double cover of $\mathbb{P}^1$ and there are two branch cuts, correspond to the two roots of

$$z^2(z-1)^2 u_2(z) = -(z-1)m_1^2 + zm_2^2 + z(z-1)m_3^2 \tag{3.8}$$

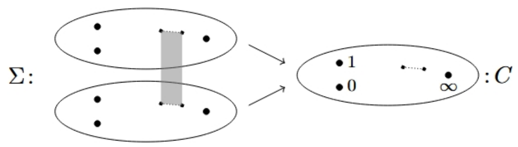

Figure 3.

using the Riemann-Hurwitz theorem, Σ is topologically a sphere.

Contour integrals of λ give integer linear combinations of residues of $\lambda = xdz = \pm\sqrt{u_2(z)}dz$ at the poles $z = 0, 1, \infty$, by construction, these residues are $\pm m_1, \pm m_2, \pm m_3$. So masses of BPS states take the form

$$Z = f_1 m_1 + f_2 m_2 + f_3 m_3 \quad f_1, f_2, f_3 \in \mathbb{Z} \tag{3.9}$$

The period computation gives the possible central charges compatible with the flavor charge lattice. The actual elementary BPS states of the free T_2 theory are the trifundamental half-hypermultiplet states with masses $\pm m_1 \pm m_2 \pm m_3$, this imposes the further restriction that $f_1 = f_2 = f_3$ mod2. This BPS masses can also be understand in brane perspective, for M2 brane that $D \subset T^*C$ and $\partial D \subset \Sigma$. Choose D such that ∂D is a contour from one branch point to the other (on one sheet) and back via the other sheet , the λ period integral gives the mass $\pm m_1 \pm m_2 \pm m_3$ (this is because doing this contour equals to go around the branch cut at one sheet, which divides the 6 poles into 2×3, and poles on different sheet contributes $\pm m_i$). Choose D to be a small circle around one of the punctures, times the interval connecting the two sheets of Σ, its boundary is a pair of circles picking up twice the same residue, giving the mass to be $2m_i$.

3.2 4d $\mathcal{N} = 2$ $SU(2)$ $N_f = 4$ theory

We consider $\mathbb{P}^1$ with four punctures

$$z_1 = 0 \; z_2 = q \; z_3 = 1 \; z_4 = \infty \tag{3.10}$$

We can consider degenerations where $q \to 0, 1, \infty$, and focus on $q \to 0$ since these three are up to permutations. We expect on general grounds that the 4d theory consists of two tinkertoys T_2 and one $SU(2)$ vector multiplet gauging an $SU(2)$ flavor symmetry of each tinkertoy.

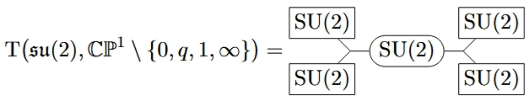

Figure 4.

Here the junction denotes the tinkertoy T_2, and four hypermultiplets transforming as two doublet representations of the $SU(2)$ gauge group. This is a $SU(2)$ theory with $N_f = 4$ flavors, each tinkertoy after gauge has a $Spin(4)$ flavor symmetry, when we look in the 4d persepective where C shrinks to a point, then the total flavor symmetry is enhanced to $Spin(8)$ since the all flavors are equal, which is bigger then $Spin(4)^2$.

3.2.1 Coulomb Branch

The Coulomb branch is determined by $\phi_2(z)$ which is required to be holomorphic and has expected poles. We still follows a similar approach in (3.6) with identifying m_1, m_2, m_3, m_4 at $0, q, 1, \infty$, but first consider the massless case, when $\phi_2(z)$ has order-1 poles, which is generally $\phi_2(z) = \frac{f(z)}{z(z-q)(z-1)} dz^2$. By using the Liouville theorem, $f(z)$ can only be a constant u. Then consider the massive case same as (3.6), we determine

$$\phi_2(z) = \frac{\frac{q}{z} m_1^2 + \frac{q(q-1)}{z-q} m_2^2 + \frac{z-q}{z-1} m_3^2 + z m_4^2 - u}{z(z-q)(z-1)} \tag{3.11}$$

for $|q|, |z| << 1$ at fixed masses, which corresponds to moving towards one side of pants, this behaves as

$$\phi_2(z) \sim \frac{-\frac{q}{z} m_1^2 + \frac{q}{z-q} m_2^2 + u}{z(z-q)} dz^2 \tag{3.12}$$

which is (3.7) for masses are m_1^2, m_2^2, u, similarly when $|q| << |z|, 1$, this behaves as the three punctured sphere whose masses are u, m_3^2, m_4^2. Since masses are background values of vector multiplet scalars, and u appears as we move towards both pants, we can identify $u = \frac{1}{2} \left\langle \mathrm{Tr} \phi^2 \right\rangle$.

3.2.2 Seiberg-Witten curve

The SW curve is defined as $\Sigma = \{x^2 = u_2(z)\}$ which is a ramified double cover over $\mathbb{P}^1$, there are four branch points by solving $u_2(z) = 0$. By Riemann-Hurwitz theorem, this is topologically a torus

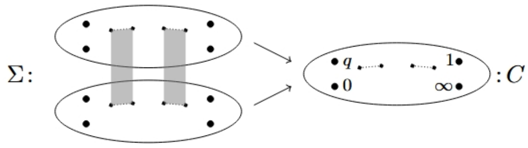

Figure 5.

For generic point at the Coulomb branch, the effective theory is a $U(1)$ theory of $\mathcal{N} = 2$ vector multiplets. But for some value of u the branch points may collide resulting 1-cycles shrinks to a point, then massless BPS particles appears, this is when the discriminant Δ_P of the polynomial $P(z) = z^2(z-q)^2(z-1)^2 u_2(z)$ is zero[5]. Δ_P is generically a degree 6 polynomial so we should expect 6 singularities on the Coulomb branch.

There are four of six singularities can be seen in $q \to 0$ limit, which is $\sqrt{u} = \pm m_1 \pm m_2$ or $\sqrt{u} = \pm m_3 \pm m_4$. Which is

$$u = (m_1 \pm m_2)^2 \quad (m_3 \pm m_4)^2 \tag{3.13}$$

the remaining two points are not so easy to determine from 3.6, correspond to the collision of branch points that sit in different pair of pants in our decomposition above[6]

$$u = \pm 2\sqrt{q(m_2^2 - m_1^2)(m_3^2 - m_4^2)} + O(q) \tag{3.14}$$

from the point of view of $N_f = 4$ SQCD, the four doublet hypermultiplets have mass parameter

$$m_1 \pm m_2 \quad m_3 \pm m_4 \tag{3.15}$$

so when the VEV of ϕ matches one of these we get a massless hypermultiplet, it has the charge ± 1 under the low-energy $U(1)$ because that is how the diagonal $U(1) \subset SU(2)$ acts on a doublet. At low energy $|u| << |m_i|, |m_i \pm m_j|$, all hypermultiplets are massive and we are left with pure SYM. If we tune m, q moreover u, we may let more than two branch points collide together and this makes more than one BPS state become massless, this may get nontrivial Argyres–Douglas SCFT at IR.

Another thing worth mentioning is the S-duality, coming from the three different pants decomposition of $\mathbb{P}^1 - \{0, q, 1, \infty\}$, these description are identical except for permutations of masses and changing $q \to 1/q$ or $q \to 1-q$.

[5]The discriminant of a degree d polynomial $P(z) = p_d \prod_{a=1}^{d}(z - z_a)$ is defined as $\Delta_P = p_d^{2d-2} \prod_{a<b}(z_a - z_b)^2$, this vanishes by construction exactly when $P(z)$ has double roots.

[6]The derivation is at the appendix

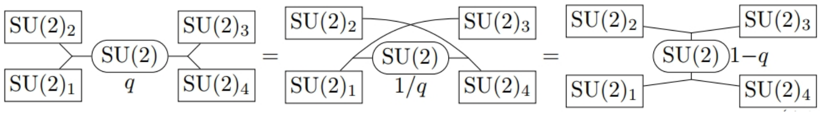

Figure 6.

While these equalities are manifest in the 6d perspective they hide deep non-perturbative physics, as they are equalities between QFTs involving completely different elementary gauge fields and matter fields.

3.3 Generalized SU(2) quivers

We first consider the five-puncture sphere $C = \mathbb{P}^1 - \{0, z_1, z_2, 1, \infty\}$, for any decomposition into three-punctured spheres the theory is like:

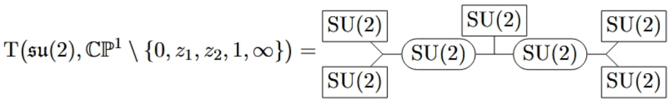

Figure 7.

In this theory, the flavor symmetry is $SU(2)^5$, not as enhanced in the $N_f = 4$ case, according to the gluing rule (2.23) we mentioned, the gluing parameters are z_1/z_2 and z_2. Their are twelve hypermultiplets and two gauge groups $SU(2)^2$ together with five flavor groups $SU(2)^5$ now, 4 of the hypermultiplets transform in $(\mathbf{2}, \mathbf{1})$ with flavor symmetry $(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, 4 of them in $(\mathbf{2}, \mathbf{2})$ with flavor symmetry $(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1})$ and last 4 of them in $(\mathbf{1}, \mathbf{2})$ with $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$.

Another thing worth mentioning is the S duality of such theory, when one of the two gauge groups is weak coupled, the theory can be seen as a $SU(2)$ $N_f = 4$ theory of one node, since we know the S duality of $N_f = 4$ theory corresponds to the different pants decomposition of four punctured spheres, now the S duality of one node corresponds to different pants decomposition of five punctured spheres. Also, since the two gauge nodes are connected by a pant in general, their S duality does not commute.

Next we generally consider the n-punctured sphere $\mathbb{P}^1 - \{z_0, \cdots, z_{n-1}\}$ with $z_0 = 0, z_{n-2} = 1, z_{n-1} = \infty$. Generally, we can assume $u_2(z)$ as

$$\sum_{i=0}^{n-2} \left[\frac{m_i^2}{(z - z_i)^2} + \frac{c_i}{z - z_i} \right] \tag{3.16}$$

with all finite puncture condition used. When applying $z \to \infty$ we get the constrains

$$\sum_{i=0}^{n-2} c_i = 0 \quad \sum_{i=0}^{n-2} m_i^2 + \sum_{i=0}^{n-2} c_i z_i = m_{n-1}^2 \tag{3.17}$$

the total number of free parameters is $(n-1)-2 = n-3$, indicating $\mathcal{B} = \mathbb{C}^{n-3}$, and the theory is a $SU(2)^{n-3}$ quiver. The Coulomb branch is parametrized by $u_i = \frac{1}{2}\text{Tr}\phi_i^2$, where ϕ_i is the scalar in i-th vector multiplet. It should be noted that for $n \geq 6$, the pants decomposition actually can have different topologies (straight on one line or like a tree, etc.) .

We can also discuss punctured $\Sigma_{g,n}$ surface a bit, generally

$$\begin{aligned} \#pants &= 2g - 2 + n \\ \#tubes &= 3g - 3 + n \end{aligned} \tag{3.18}$$

for a n-punctured torus, we have n T_2 and $SU(2)^n$ gauge quiver theory with a circular quiver. Since torus has a modular property, we can set $u_2(z)$ as a elliptic function, generated from Weierstrass elliptic function and zeta function by

$$u_2(z) = u_0 + \sum_{a=1}^{n} m_a^2 \wp(z - z_a) + \sum_{a=1}^{n} c_a \zeta(z - z_a) \tag{3.19}$$

since zeta is quasi elliptic, we require $\sum_a c_a = 0$. The total number of free parameters is $1 + (n-1) = n$, matched with $SU(2)^n$.

3.4 Linear quiver $\mathfrak{su}(N)$ theories

We move on to $\mathfrak{su}(N)$ class S theories, the tinkertoys are slightly different here, which we can see in the example of $SU(N)$ $N_f = 2N$ theory as a generation of $SU(2)$ $N_f = 4$ theories. The flavor symmetries now is $\mathfrak{u}(2N)$ in general, if we expect a similar tinkertoy picture as $N_f = 4$ case, we decompose the $2N$ hyper multiplet with $2 \times N$ with each has $\mathfrak{u}(N) = \mathfrak{u}(1) \times \mathfrak{su}(N)$ flavor symmetry, the picture is

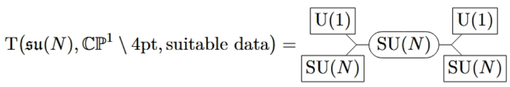

Figure 8.

As we have mentioned, the $\mathfrak{u}(2)$ is enhanced to $\mathfrak{so}(4)$ as $N = 2$, which corresponds $\mathfrak{u}(1)$ is enhanced to $\mathfrak{su}(2)$. With this picture, we have two types of punctures, with $SU(N)$ and $U(1)$ flavor symmetry respectively (named as 'full' and 'simple' below), and each tinkertoy with $U(1) \times SU(N)^2$.

For the full punctures, we learn from $SU(2)$ that they are encoded in

$$\varphi(z) \sim \left(\frac{m_i}{z - z_i} + O(1)\right) dz \Longrightarrow \phi_k(z) = \left(\frac{(-1)^{k+1}\sigma_k(m_i)}{(z - z_i)^k} + O\left(\frac{1}{(z - z_i)^{k-1}}\right)\right) dz^k \tag{3.20}$$

for $m_i \in \mathfrak{su}(N)$ and the symmetric polynomials $\sigma_k(m_i)$ are defined by

$$\det(X - m_i) = X^N + \sum_{k\geq 2} X^{N-k}(-1)^k \sigma_k(m_i) \tag{3.21}$$

at the massless limit, full and simple punctures also has different behavior

$$\begin{aligned} &full: && \phi_k(z) = O\left(\frac{1}{((z-z_i)^{k-1})}\right) dz^k \\ &simple: && \phi_k(z) = O\left(\frac{1}{z-z_i}\right) dz^k \end{aligned} \tag{3.22}$$

We can now go to determine the $\phi_k(z)$ for these tinkertoys, first we consider the massless case, we set $z = 1$ for the simple pole and $z = 0, \infty$ for the order $k-1$ pole. For $z = 0, 1$, this gives

$$u_k(z) = \frac{f(z)}{z^{k-1}(z-1)} \tag{3.23}$$

if $f(z)$ has degree d, $u_k(z) \sim z^{d-k}$, however $z^{d-k}dz^k$ becomes $w^{-k-d}dw^k$ when $w = 1/z$, for $w = 0$ we let $-k-d \geq -(k-1)$ which gives $d \leq -1$, then ϕ_k can only be 0 for all $k = 2, \cdots, N$, the Coulomb branch do not has a free parameter. For the massive case, we consider $N = 3$, near the simple puncture this behaves as

$$\varphi(z) \sim \left(\frac{m}{z-1} + p\right) dz \tag{3.24}$$

where $m = \text{diag}(2\mu, -\mu, -\mu)$, we can get

$$\begin{aligned} u_2 &= \frac{3\mu^2}{(z-1)^2} + \frac{\text{Tr}(mp)}{z-1} + O(1) \\ u_3 &= \frac{2\mu^3}{(z-1)^3} + \frac{\mu\text{Tr}(mp)}{(z-1)^2} + O(\frac{1}{z-1}) \end{aligned} \tag{3.25}$$

form $\det(x - \varphi_z)$, since all these coefficients are composed from simple puncture data, we can also deduce there's no Coulomb branch also when it's massive.

Before moving to quiver theories, we can also briefly mention the generalization of T_2. For the A_{N-1} theory, the basic three-punctured sphere with three full punctures is called T_N. Namely, T_N is the class S theory obtained by compactifying the 6d $\mathcal{N} = (2,0)$ theory of type $\mathfrak{su}(N)$ on a sphere with three full punctures. Since each full puncture carries an $SU(N)$ flavor symmetry, T_N has at least

$$SU(N)_1 \times SU(N)_2 \times SU(N)_3 \tag{3.26}$$

flavor symmetry. It is the natural higher rank analogue of the trifundamental tinkertoy T_2.

For $N = 2$, this theory is just the T_2 theory we discussed above, which is equivalent to a trifundamental half-hypermultiplet in

$$(\mathbf{2}, \mathbf{2}, \mathbf{2}) \tag{3.27}$$

of $SU(2)^3$, or equivalently four free hypermultiplets. However, this free-field description is special to $N = 2$. For $N = 3$, T_3 is already an interacting strongly coupled SCFT, known as the Minahan–Nemeschansky E_6 theory. In this case, the manifest $SU(3)^3$ flavor symmetry from the three punctures is enhanced to E_6. For general $N \geq 3$, T_N is a non-Lagrangian $\mathcal{N} = 2$ SCFT in general.

The Coulomb branch of T_N can be read from the Hitchin differentials on the three-punctured sphere. For three full punctures, the number of Coulomb branch operators of scaling dimension k is

$$d_k = k - 2, \qquad k = 3, \cdots, N. \tag{3.28}$$

Thus

$$\dim_{\mathbb{C}} \mathcal{B}(T_N) = \sum_{k=3}^{N} (k-2) = \frac{(N-1)(N-2)}{2}. \tag{3.29}$$

This also agrees with the special case T_2, which has no Coulomb branch.

Therefore, in higher rank class S theories, the role played by the free trifundamental matter T_2 is replaced by T_N. Gluing two full punctures means gauging the diagonal subgroup

$$SU(N)_{\text{diag}} \subset SU(N) \times SU(N) \tag{3.30}$$

by a 4d $\mathcal{N} = 2$ vector multiplet. In this sense, general class S theories can be constructed from T_N blocks and vector multiplets, but unlike the $SU(2)$ case, the elementary building blocks are generically already strongly coupled SCFTs.

Now can consider linear quiver gauge theory by connecting these tinkertoys mentioned at beginning

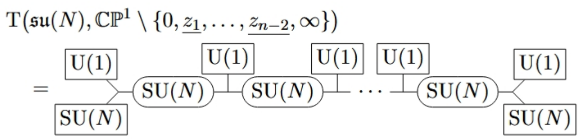

Figure 9.

we can consider the ϕ_k of this model with $2\ k - 1$ poles at $0, \infty$ and simple poles at $z_1, \cdots, z_{n-3}, 1$. We set the ansatz as

$$u_k(z) = \frac{P^{(k)}(z)}{z^{k-1}(z-z_1)\cdots(z-z_{n-3})(z-1)} \sim z^{d-(n+k-3)} \tag{3.31}$$

for $P^{(k)}$ has degree d. Since $d - (n + k - 3) \leq -(k + 1)$, we have $d \leq n - 4$, which implies $P^{(k)}$ has at most $n - 3$ parameters. The Coulomb branch dimension is (for $k = 2, \cdots, N$ has $N - 1$ parameters)

$$\dim_{\mathbb{C}} \mathcal{B} = (n-3)(N-1) \tag{3.32}$$

for the degeneration limit of this Seiberg-Witten curve, where z_{i+1}/z_i is fixed except one $z_{j+1}/z_j \to +\infty$ for some j. In this limit, the "left side" poles $0, z_1, \cdots, z_j$ stays while the "right side" goes to infinity. Overall this gives a new surface $\mathbb{P}^1 - \{0, z_1, \cdots, z_j, \infty_{new}\}$ with new boundary condition at ∞, this new puncture is actually a new full puncture.

A Appendix

A.1 Regular punctures and Young diagrams

In this appendix, we briefly explain the relation between regular punctures, tame singularities and Young diagrams in class S theories of type A_{N-1}. Recall that the Coulomb branch of the compactified theory is described by a Hitchin system on the punctured Riemann surface C. The main object is the Hitchin field

$$\Phi_z dz, \tag{A.1}$$

where Φ_z is valued in the Lie algebra $\mathfrak{sl}(N)$.

Let us consider a puncture at $z = 0$. A regular, or tame, puncture means that the Hitchin field is allowed to have at most a simple pole at the puncture:

$$\Phi_z = \frac{A}{z} + \text{regular.} \tag{A.2}$$

Here $A \in \mathfrak{sl}(N)$ is called the residue of the Hitchin field at the puncture. If higher order poles are allowed, for example

$$\Phi_z = \frac{A_2}{z^2} + \frac{A_1}{z} + \cdots, \tag{A.3}$$

then the puncture is irregular, or wild. Therefore the word “regular” or “tame” refers to the order of the pole of Φ_z.

However, even if we restrict to regular punctures, there are still many different types of punctures. The difference is encoded in the possible form of the residue A. For the A_{N-1} theory, regular punctures are classified by partitions of N, or equivalently by Young diagrams with N boxes:

$$N = n_1 + n_2 + \cdots + n_\ell. \tag{A.4}$$

We denote the corresponding Young diagram by

$$Y = [n_1, n_2, \cdots, n_\ell]. \tag{A.5}$$

Intuitively, this Young diagram describes how the eigenvalues of the residue A are grouped, or equivalently how the N sheets of the spectral curve degenerate near the puncture.

The most important example is the full puncture. It corresponds to the partition

$$[1^N] = [\underbrace{1, 1, \cdots, 1}_{N \text{ times}}]. \tag{A.6}$$

In this case the residue can be taken to be a generic semisimple matrix

$$A = \mathrm{diag}(m_1, m_2, \cdots, m_N), \qquad \sum_{i=1}^{N} m_i = 0. \tag{A.7}$$

Thus there are $N-1$ independent mass parameters. The flavor symmetry associated to a full puncture is

$$SU(N). \tag{A.8}$$

This is why it is called a full puncture: it carries the maximal flavor symmetry.

Another important example is the simple puncture, corresponding to the partition

$$[N-1, 1]. \tag{A.9}$$

In this case the residue is more degenerate. A typical representative is

$$A = \mathrm{diag}\big((N-1)m, -m, \cdots, -m\big), \tag{A.10}$$

which again has vanishing trace. There is only one independent mass parameter, and the associated flavor symmetry is

$$U(1). \tag{A.11}$$

Thus a simple puncture is the minimal nontrivial regular puncture.

The flavor symmetry can be read from the Young diagram as follows. Suppose that rows of length h appear with multiplicity m_h. Then the flavor symmetry is roughly

$$S\left(\prod_h U(m_h)\right), \tag{A.12}$$

up to possible special enhancements. For example,

$$[1^N] \quad \Rightarrow \quad S(U(N)) = SU(N), \tag{A.13}$$

while

$$[N-1, 1] \quad \Rightarrow \quad S(U(1) \times U(1)) = U(1). \tag{A.14}$$

The same Young diagram also determines the allowed pole orders of the Hitchin differentials. The Seiberg–Witten curve of the A_{N-1} theory can be written as

$$x^N + \sum_{k=2}^{N} \phi_k(z) x^{N-k} = 0, \tag{A.15}$$

where

$$\phi_k(z) = \mathrm{Tr}\, \Phi_z^k \tag{A.16}$$

is a meromorphic k-differential on C. Near a puncture, we may write

$$\phi_k(z) \sim \frac{1}{z^{p_k}} + \cdots . \tag{A.17}$$

The integers p_k are determined by the Young diagram of the puncture.

For the purpose of counting Coulomb branch parameters, the two most useful cases are

puncture	partition	flavor symmetry	pole order for Coulomb data
full	$[1^N]$	$SU(N)$	$p_k = k-1$
simple	$[N-1,1]$	$U(1)$	$p_k = 1$
trivial	$[N]$	none	$p_k = 0$

(A.18)

Here $k = 2, \cdots, N$. This convention is especially useful when we count the dimension of the Coulomb branch by counting the allowed meromorphic differentials.

There is a small subtlety in this statement. Since a regular puncture has

$$\Phi_z \sim \frac{A}{z}, \tag{A.19}$$

one might expect

$$\phi_k = \mathrm{Tr}\, \Phi_z^k \sim \frac{\mathrm{Tr} A^k}{z^k}. \tag{A.20}$$

Thus ϕ_k can appear to have a pole of order k. This is not in contradiction with the table above. The highest order singular terms are fixed by the mass parameters, while the table records the pole orders relevant for the unfixed Coulomb branch data. In other words, the mass deformations determine the leading singular behavior, whereas the Coulomb moduli are encoded in the remaining allowed meromorphic coefficients.

Let us spell out the simplest examples. For the A_1 theory, namely $N = 2$, the only nontrivial regular puncture is the full puncture

$$[1,1] \tag{A.21}$$

which carries an $SU(2)$ flavor symmetry. This is why in the $SU(2)$ class S examples, each regular puncture naturally gives an $SU(2)$ flavor symmetry. For example, the four-punctured sphere describing the $SU(2)$ theory with four flavors has four full punctures and hence a manifest

$$SU(2)^4 \tag{A.22}$$

flavor symmetry.

For the A_2 theory, namely $N = 3$, there are three types of regular punctures:

$$[1,1,1], \qquad [2,1], \qquad [3] \tag{A.23}$$

They correspond respectively to a full puncture, a simple puncture and a trivial puncture:

partition	puncture	flavor symmetry
[1,1,1]	full	SU(3)
[2,1]	simple	U(1)
[3]	trivial	none

(A.24)

For example, a full puncture has a generic residue

$$A_{\text{full}} = \text{diag}(m_1, m_2, m_3), \qquad m_1 + m_2 + m_3 = 0, \tag{A.25}$$

while a simple puncture has a degenerate residue of the form

$$A_{\text{simple}} = \text{diag}(2m, -m, -m). \tag{A.26}$$

In summary, regular or tame describes the order of the singularity of the Hitchin field:

$$\text{regular/tame puncture} \quad \Longleftrightarrow \quad \Phi_z \text{ has at most a simple pole.} \tag{A.27}$$

The Young diagram describes the type of the residue at this simple pole:

$$\text{Young diagram} \quad \Longleftrightarrow \quad \text{degeneration type of the residue } A. \tag{A.28}$$

It determines both the flavor symmetry of the puncture and the pole structure of the differentials ϕ_k. In particular,

$$[1^N] \Rightarrow SU(N) \text{ full puncture}, \qquad [N-1,1] \Rightarrow U(1) \text{ simple puncture}, \qquad [N] \Rightarrow \text{trivial puncture}. \tag{A.29}$$

A.2 Derivation of (3.14)

The Seiberg–Witten curve is

$$x^2 = u_2(z), \tag{A.30}$$

so branch points are zeros of $u_2(z)$.

To remove denominators, define

$$P(z) := z^2(z-q)^2(z-1)^2 u_2(z). \tag{A.31}$$

Branch point collision means that $P(z)$ has a double root:[7]

$$P(z) = 0, \qquad P'(z) = 0. \tag{A.32}$$

Using equation (3.11), one obtains

$$\begin{aligned} P(z) = {} & q m_1^2 (z-q)(z-1) + q(q-1) m_2^2 z(z-1) \\ & + m_3^2 z(z-q)^2 + m_4^2 z^2 (z-q)(z-1) - uz(z-q)(z-1). \end{aligned} \tag{A.33}$$

The two singularities of interest lie in the scaling region

$$q = \epsilon^2, \qquad z = c\epsilon, \qquad u = d\epsilon. \tag{A.34}$$

[7]The solution of this have three part: 1. $z \sim 0, z \sim q$ gives $\sqrt{u} \sim \pm m_1 \pm m_2$. 2.$z \sim 1, z \sim q$ gives $\sqrt{u} \sim \pm m_3 \pm m_4$. The third is $q = \epsilon^2, z = c\epsilon, u = d\epsilon$ discussed below

Keeping only the leading term gives

$$P(z) = \epsilon^3 \left[c(m_2^2 - m_1^2) + c^3(m_3^2 - m_4^2) + dc^2\right] + O(\epsilon^4). \tag{A.35}$$

Hence the double-root condition becomes

$$m_2^2 - m_1^2 + dc + (m_3^2 - m_4^2)c^2 = 0, \qquad d + 2(m_3^2 - m_4^2)c = 0. \tag{A.36}$$

Solving these equations gives

$$c^2 = \frac{m_2^2 - m_1^2}{m_3^2 - m_4^2}, \qquad d = \mp 2\sqrt{(m_2^2 - m_1^2)(m_3^2 - m_4^2)}. \tag{A.37}$$

Since $u = d\sqrt{q}$, the remaining two singularities are

$$u = \pm 2\sqrt{q(m_2^2 - m_1^2)(m_3^2 - m_4^2)} + O(q). \tag{A.38}$$

Acknowledgments

References

[1] B. Le Floch, J. Phys. A **55**, no.35, 353002 (2022) doi:10.1088/1751-8121/ac5945 [arXiv:2006.14025 [hep-th]].

[2] N. J. Hitchin, Duke Math. J. **54**, 91-114 (1987) doi:10.1215/S0012-7094-87-05408-1